

THE STRUCTURES OF STANDARD (\mathfrak{g}, K) -MODULES OF $SL(3, \mathbf{R})$.

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ABSTRACT. We describe explicitly the structures of standard (\mathfrak{g}, K) -modules of $SL(3, \mathbf{R})$.

1. INTRODUCTION

As far as we know, for some ‘small’ semisimple Lie groups G , the (\mathfrak{g}, K) -module structures of standard representations are completely described. For example, the description of them for $SL(2, \mathbf{R})$ is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. $SU(n, 1)$ in [1] and $Spin(1, 2n)$ in [6]. However, for Lie groups of higher rank, there are few references as far as the author knows. It seems to be difficult to describe the whole (\mathfrak{g}, K) -module structures even for standard representations of classical groups of higher rank, since their K -types are not multiplicity free. In the papers [4] and [5], the (\mathfrak{g}, K) -module structures of some standard representations of $Sp(2, \mathbf{R})$ are described by T. Oda. In the former paper [3], we extend the result for principal series representations of $Sp(3, \mathbf{R})$. The method in these papers is applicable to study of standard representations of another groups. In this paper, we use this method to study standard (\mathfrak{g}, K) -modules of $SL(3, \mathbf{R})$.

Before describing the case of $SL(3, \mathbf{R})$, let us explain the problem in a more precise form for a general real semisimple Lie group G with its Lie algebra \mathfrak{g} . Fix a maximal compact subgroup K of G . Since any standard (\mathfrak{g}, K) -modules are realized as subspaces of $L^2(K)$ as K -modules, we investigate the K -module structure of standard (\mathfrak{g}, K) -modules by the Peter-Weyl’s theorem. In order to describe the action of \mathfrak{g} or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of \mathfrak{p} or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Therefore, the investigation of the action of \mathfrak{p} or $\mathfrak{p}_{\mathbf{C}}$ is essential to give the description of the (\mathfrak{g}, K) -module structure of a standard representation. To study the action of $\mathfrak{p}_{\mathbf{C}}$, we compute the linear map $\Gamma_{\tau, i}$ defined as follows. Let (π, H_{π}) be a standard representation of G with its subspace $H_{\pi, K}$ of K -finite vectors. For a K -type (τ, V_{τ}) of π , and a nonzero K -homomorphism $\eta: V_{\lambda} \rightarrow H_{\pi, K}$, we define a linear map $\tilde{\eta}: \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_{\lambda} \rightarrow H_{\pi, K}$ by $X \otimes v \mapsto X \cdot \eta(v)$. Then $\tilde{\eta}$ is a K -homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K . Let $V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq \bigoplus_{i \in I} V_{\tau_i}$ be the decomposition into a direct sum of irreducible K -modules and ι_i an injective K -homomorphism from V_{τ_i} to $V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$ for each i . We define a linear map $\Gamma_{\tau, i}: \text{Hom}_K(V_{\tau}, H_{\pi, K}) \rightarrow \text{Hom}_K(V_{\tau_i}, H_{\pi, K})$ by $\eta \mapsto \tilde{\eta} \circ \iota_i$. These linear maps $\Gamma_{\tau, i}$ ($i \in I$) characterize the action of $\mathfrak{p}_{\mathbf{C}}$. Our purpose of this paper is to give explicit expressions of ι_i and $\Gamma_{\tau, i}$ when π is a P -principal series representation of $G = SL(3, \mathbf{R})$ for each standard parabolic subgroup P of G . As a result, we obtain infinite number of ‘contiguous relations’, a kind of system of differential-difference relations among vectors in $H_{\pi}[\tau]$ and $H_{\pi}[\tau_i]$. Here $H_{\pi}[\tau]$ is τ -isotypic component of H_{π} . These are described in Proposition 4.2, Theorem 5.5 and 6.5.

As an application, we can utilize the contiguous relations to obtain the explicit formulae of some spherical functions. In the paper [2], H. Manabe, T. Ishii and T. Oda give the explicit formulae of Whittaker functions of principal series representations of $SL(3, \mathbf{R})$ to solve the holonomic system of differential equations characterizing those functions, which is derived from the Capelli elements and the contiguous relations around minimal K -type. We can obtain the holonomic systems characterizing Whittaker functions of generalized principal series representations of $SL(3, \mathbf{R})$ from the result of this paper. We hope that this interesting possibility will be considered in future work. On the other hand, if we have the explicit formula

of Whittaker function with a certain K -type, then we can give those with another K -type by using contiguous relations.

We give the contents of this paper. In Section 2, we recall the classical case $SL(2, \mathbf{R})$ shortly. In Section 3, we recall the structure of $SL(3, \mathbf{R})$ and define a standard representations obtained by a parabolic induction with respect to the standard parabolic subgroups. In Section 4, we introduce the standard basis of a finite dimensional irreducible representation of K and give explicit expressions of $\iota_i: V_{\tau_i} \rightarrow V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$. In Section 5, we introduce the general setting of this paper and give matrix representations of $\Gamma_{\tau,i}$ for principal series representations in Theorem 5.5. In Section 6, we give the matrix representations of $\Gamma_{\tau,i}$ for generalized principal series representations in Theorem 6.5. In Section 7, we give explicit expressions of the action of $\mathfrak{p}_{\mathbf{C}}$ in Proposition 7.2.

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2. THE STANDARD (\mathfrak{g}, K) -MODULES OF $SL(2, \mathbf{R})$

We start with a short review of the most classical case, i.e. the case of the group $SL(2, \mathbf{R})$.

2.1. The principal series representations of $SL(2, \mathbf{R})$. We denote by \mathbf{Z} , \mathbf{R} and \mathbf{C} the ring of rational integers, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq 0}$ be the set of non-negative integers, 1_n be the unit matrix in the space $M_n(\mathbf{R})$ of real matrices of size n and $O_{m,n}$ be the zero matrix of size $m \times n$. We denote by δ_{ij} the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For a Lie algebra \mathfrak{l} , we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of \mathfrak{l} .

We put

$$\begin{aligned} G' &= SL(2, \mathbf{R}), \quad M' = \{m = \text{diag}(\varepsilon, \varepsilon^{-1}) \mid \varepsilon \in \{\pm 1\}\}, \quad A' = \{a(r) = \text{diag}(r, r^{-1}) \mid r \in \mathbf{R}_{>0}\}, \\ N' &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbf{R} \right\}, \quad K' = SO(2) = \left\{ \kappa_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid t \in \mathbf{R} \right\}. \end{aligned}$$

Let \mathfrak{g}' , \mathfrak{k}' , \mathfrak{a}' and \mathfrak{n}' be Lie algebras of G' , K' , A' and N' , respectively.

For $\nu \in \mathbf{C}$ and a character σ of M' , the principal series representation $\pi_{(\nu, \sigma)}$ of G' is defined as the right regular representation of G' on the space $H_{(\nu, \sigma)}$ which is the completion of

$$H_{(\nu, \sigma)}^{\infty} = \left\{ f: G' \rightarrow \mathbf{C} \text{ smooth} \mid \begin{array}{l} f(namx) = r^{\nu+1} \sigma(m) f(x) \\ \text{for } n \in N', a = a(r) \in A', m \in M', x \in G' \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_{K'} |f(k)|^2 dk.$$

The restriction map $r_{K'}: H_{(\nu, \sigma)} \ni f \mapsto f|_{K'} \in L^2(K')$ is an injective K' -homomorphism when $L^2(K')$ is endowed with right regular action of K' . Then the image of $r_{K'}$ is the following subspace of $L^2(K')$:

$$L_{(M', \sigma)}^2(K') = \{f \in L^2(K') \mid f(mx) = \sigma(m)f(x) \text{ for a.e. } m \in M', x \in K'\}.$$

We have an irreducible decomposition of the K' -module $L^2(K')$:

$$L^2(K') = \widehat{\bigoplus_{p \in \mathbf{Z}} \mathbf{C} \cdot \tilde{\chi}_p},$$

where $\tilde{\chi}_p: K' \ni \kappa_t \mapsto e^{\sqrt{-1}pt} \in \mathbf{C}^{\times}$.

Therefore we have an isomorphism

$$H_{(\nu, \sigma)} \rightarrow L_{(M', \sigma)}^2(K') = \begin{cases} \widehat{\bigoplus_{p \in 2\mathbf{Z}} \mathbf{C} \cdot \tilde{\chi}_p}, & \text{if } \sigma(-1_2) = 1, \\ \widehat{\bigoplus_{p \in 1+2\mathbf{Z}} \mathbf{C} \cdot \tilde{\chi}_p}, & \text{if } \sigma(-1_2) = -1. \end{cases}$$

Let $\chi_p \in H_{(\nu, \sigma)}$ be an inverse image of $\tilde{\chi}_p$ by this isomorphism.

Now we take a basis $\{w, x_+, x_-\}$ of $\mathfrak{g}'_{\mathbf{C}}$ defined by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_{\pm} = \begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix}.$$

Here we note that

$$\mathfrak{g}'_{\mathbf{C}} = \mathfrak{k}'_{\mathbf{C}} \oplus \mathfrak{p}'_{\mathbf{C}}, \quad \mathfrak{k}'_{\mathbf{C}} = \mathbf{C} \cdot w, \quad \mathfrak{p}'_{\mathbf{C}} = \mathbf{C} \cdot x_+ \oplus \mathbf{C} \cdot x_-.$$

is a complexification of a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ with respect to a Cartan involution $\mathfrak{g}' \ni X \mapsto -{}^t X \in \mathfrak{g}'$ where ${}^t X$ means transpose of X .

Since $w \in \mathfrak{k}'$, we see that

$$(2.1) \quad \pi_{(\nu, \sigma)}(w)\chi_p = \sqrt{-1}p\chi_p$$

from direct computation. Here we denote the differential of $\pi_{(\nu, \sigma)}$ again by $\pi_{(\nu, \sigma)}$. The action of $\mathfrak{p}'_{\mathbf{C}}$ is given in the following proposition.

Proposition 2.1. $\pi_{(\nu, \sigma)}(x_{\pm})\chi_p = (\nu + 1 \pm p)\chi_{p \pm 2}$.

Proof. By the relations

$$[w, x_{\pm}] = \pm 2\sqrt{-1}x_{\pm},$$

we have

$$(2.2) \quad \pi_{(\nu, \sigma)}(w)(\pi_{(\nu, \sigma)}(x_{\pm})\chi_p) = \sqrt{-1}(p \pm 2)(\pi_{(\nu, \sigma)}(x_{\pm})\chi_p).$$

Here $[\cdot, \cdot]$ is the bracket product. From the equations (2.1) and (2.2), we see that $\pi_{(\nu, \sigma)}(x_{\pm})\chi_p \in \mathbf{C} \cdot \chi_{p \pm 2}$.

The elements x_{\pm} of $\mathfrak{p}'_{\mathbf{C}}$ have the following expressions according to Iwasawa decomposition $\mathfrak{g}'_{\mathbf{C}} = \mathfrak{n}'_{\mathbf{C}} \oplus \mathfrak{a}'_{\mathbf{C}} \oplus \mathfrak{k}'_{\mathbf{C}}$:

$$x_{\pm} = \pm 2\sqrt{-1}E' + H' \mp \sqrt{-1}w$$

where $E' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}'_{\mathbf{C}}$ and $H' = \text{diag}(1, -1) \in \mathfrak{a}'_{\mathbf{C}}$. From this expression and the definition of the space $H_{(\nu, \sigma)}$, we have the value of $\pi_{(\nu, \sigma)}(x_{\pm})\chi_p$ at $1_2 = \kappa_0 \in K'$ as follows:

$$\begin{aligned} \pi_{(\nu, \sigma)}(x_{\pm})\chi_p(1_2) &= \pm 2\sqrt{-1}\pi_{(\nu, \sigma)}(E')\chi_p(1_2) + \pi_{(\nu, \sigma)}(H')\chi_p(1_2) \mp \sqrt{-1}\pi_{(\nu, \sigma)}(w)\chi_p(1_2) \\ &= 0 + (\nu + 1) \mp \sqrt{-1}(\sqrt{-1}p) \\ &= \nu + 1 \pm p. \end{aligned}$$

Since $\chi_{p \pm 2}(1_2) = 1$, we obtain $\pi_{(\nu, \sigma)}(x_{\pm})\chi_p = (\nu + 1 \pm p)\chi_{p \pm 2}$. \square

From this proposition, we obtain the following.

Proposition 2.2. (i) Let k be an integer such that $k \geq 2$. If $\nu = k - 1$ and $\sigma(-1) = (-1)^k$, there is an injective homomorphism from D_k^{\pm} to $\pi_{(\nu, \sigma)}$. Here D_k^+ and D_k^- are discrete series representations of $SL(2, \mathbf{R})$ with the Blattner parameter k and $-k \in \mathbf{Z}$, respectively. Moreover the quotient (\mathfrak{g}', K') -modules $\pi_{(\nu, \sigma)}/(D_k^+ \oplus D_k^-)$ is of dimension $k - 1$.

(ii) Let k be an integer such that $k \geq 2$. If $\nu = -k + 1$ and $\sigma(-1) = (-1)^k$, the $(k - 1)$ -dimensional subspace F_{k-2} of $H_{(\nu, \sigma)}$ generated by

$$\{\chi_p \mid p = -k + 2, -k + 4, \dots, k - 2\}$$

is G' -invariant and is isomorphic to the symmetric tensor representation of degree $k - 2$. Moreover the quotient $\pi_{(\nu, \sigma)}/F_{k-2}$ is isomorphic to $D_k^+ \oplus D_k^-$.

- (iii) If $\nu = 0$ and $\sigma(1_2) = -1$, $\pi_{(\nu, \sigma)}$ is a direct sum of two irreducible representations, called limit of discrete series representations.
- (iv) If (ν, σ) is not in the cases of (i), (ii) and (iii), $\pi_{(\nu, \sigma)}$ is irreducible.

We are going to show the analogue of Proposition 2.1 for $SL(3, \mathbf{R})$ in Theorem 5.5 and 6.5.

3. PRELIMINARIES

3.1. Groups and algebras. Let G be the special linear group $SL(3, \mathbf{R})$ of degree three and \mathfrak{g} be its Lie algebra. We define a Cartan involution θ of G by $G \ni g \mapsto {}^t g^{-1} \in G$. Here g^{-1} means the inverse of g . Then a maximal compact subgroup of G is given by

$$K = \{g \in G \mid \theta(g) = g\} = SO(3).$$

If we denote the differential of θ again by θ , then we have $\theta(X) = -{}^t X$ for $X \in \mathfrak{g}$. Let \mathfrak{k} and \mathfrak{p} be the $+1$ and the -1 eigenspaces of θ in \mathfrak{g} , respectively, that is,

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid {}^t X = -X\} = \mathfrak{so}(3), \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid {}^t X = X\}.$$

Then \mathfrak{k} is the Lie algebra of K and \mathfrak{g} has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Put $\mathfrak{a}_0 = \{\text{diag}(t_1, t_2, t_3) \mid t_i \in \mathbf{R} \ (1 \leq i \leq 3), \ t_1 + t_2 + t_3 = 0\}$. Then \mathfrak{a}_0 is a maximal abelian subalgebra of \mathfrak{p} . For each $1 \leq i \leq 3$, we define a linear form e_i on \mathfrak{a}_0 by $\mathfrak{a}_0 \ni \text{diag}(t_1, t_2, t_3) \mapsto t_i \in \mathbf{C}$. The set Σ of the restricted roots for $(\mathfrak{a}_0, \mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_0, \mathfrak{g}) = \{e_i - e_j \mid 1 \leq i \neq j \leq 3\}$, and the subset $\Sigma^+ = \{e_i - e_j \mid 1 \leq i < j \leq 3\}$ forms a positive root system. For each $\alpha \in \Sigma$, we denote the restricted root space by \mathfrak{g}_α and choose a restricted root vector E_α in \mathfrak{g}_α as follows:

$$E_{e_1 - e_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_1 - e_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_2 - e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $E_{-\alpha} = {}^t E_\alpha$ for $\alpha \in \Sigma^+$. If we put $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, then \mathfrak{g} has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$. Also we have $G = N_0 A_0 K$, where $N_0 = \exp(\mathfrak{n}_0)$ and $A_0 = \exp(\mathfrak{a}_0)$.

The group G has three non-trivial standard parabolic subgroups P_0, P_1, P_2 with

$$P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}, \quad P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}.$$

Let $\mathfrak{n}_1, \mathfrak{n}_2$ be subalgebras of \mathfrak{n}_0 defined by $\mathfrak{n}_1 = \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_1 - e_3}$, $\mathfrak{n}_2 = \mathfrak{g}_{e_1 - e_3} \oplus \mathfrak{g}_{e_2 - e_3}$. We take a basis $\{H_1, H_2\}$ of \mathfrak{a}_0 defined by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and set $H^{(1)} = 2H_1 - H_2$, $H^{(2)} = H_1 + H_2$. we define subalgebras $\mathfrak{a}_1, \mathfrak{a}_2$ of \mathfrak{a}_0 by $\mathfrak{a}_1 = \mathbf{R} \cdot H^{(1)}$, $\mathfrak{a}_2 = \mathbf{R} \cdot H^{(2)}$. We specify Langland decompositions of $P_i = N_i A_i M_i$ ($0 \leq i \leq 2$) by

$$\begin{aligned} M_0 &= \{\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2) \mid \varepsilon_i \in \{\pm 1\} \ (1 \leq i \leq 2)\}, \\ M_1 &= \left\{ \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix} \middle| h \in SL^\pm(2, \mathbf{R}) \right\}, \quad A_1 = \exp(\mathfrak{a}_1), \quad N_1 = \exp(\mathfrak{n}_1), \\ M_2 &= \left\{ \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix} \middle| h \in SL^\pm(2, \mathbf{R}) \right\}, \quad A_2 = \exp(\mathfrak{a}_2), \quad N_2 = \exp(\mathfrak{n}_2). \end{aligned}$$

Here $SL^\pm(2, \mathbf{R}) = \{g \in GL(2, \mathbf{R}) \mid \det(g) = \pm 1\}$. For $i = 1, 2$, let \mathfrak{m}_i be a Lie algebra of M_i .

3.2. Definition of the P_i -principal series representations of G . For $0 \leq i \leq 2$, in order to define the P_i -principal series representation of G , we prepare the data (ν_i, σ_i) as follows.

For $\nu_0 \in \text{Hom}_{\mathbf{R}}(\mathfrak{a}_0, \mathbf{C})$, we define a coordinate $(\nu_{0,1}, \nu_{0,2}) \in \mathbf{C}^2$ by $\nu_{0,i} = \nu_0(H_i)$ ($i = 1, 2$). Then the half sum $\rho_0 = \frac{1}{2} \left(\sum_{\alpha \in \Sigma_+} \alpha \right) = e_1 - e_3$ of the positive roots has coordinate $(\rho_{0,1}, \rho_{0,2}) = (2, 1)$. We define a quasicharacter $e^{\nu_0}: A_0 \rightarrow \mathbf{C}^\times$ by

$$e^{\nu_0}(a) = a_1^{\nu_{0,1}} a_2^{\nu_{0,2}}, \quad a = \text{diag}(a_1, a_2, a_3) \in A_0.$$

We fix a character σ_0 of M_0 . σ_0 is realized by $(\sigma_{0,1}, \sigma_{0,2}) \in \{0, 1\}^{\oplus 2}$ such that

$$\sigma_0(\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}}, \quad \varepsilon_1, \varepsilon_2 \in \{\pm 1\}.$$

For each $i = 1, 2$, we identify $\nu_i \in \text{Hom}_{\mathbf{R}}(\mathfrak{a}_i, \mathbf{C})$ with a complex number $\nu_i(H^{(i)}) \in \mathbf{C}$. Let ρ_i ($i = 1, 2$) be the half sums of positive roots whose root spaces are contained in \mathfrak{n}_i , i.e. $\rho_1 = \frac{1}{2}(2e_1 - e_2 - e_3)$, $\rho_2 = \frac{1}{2}(e_1 + e_2 - 2e_3)$. Then both ρ_1 and ρ_2 are identified with 3. We identify M_i ($i = 1, 2$) with $SL^\pm(2, \mathbf{R})$ by natural isomorphisms $m_i: SL^\pm(2, \mathbf{R}) \rightarrow M_i$ ($i = 1, 2$) defined by

$$m_1(h) = \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix}, \quad m_2(h) = \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix} \quad (h \in SL^\pm(2, \mathbf{R})).$$

Then we fix a discrete series representation $\sigma_i = D_k = \text{Ind}_{SL(2, \mathbf{R})}^{SL^\pm(2, \mathbf{R})}(D_k^+)$ of $M_i \simeq SL^\pm(2, \mathbf{R})$ where D_k^+ is a discrete series representation of $SL(2, \mathbf{R})$ with the Blattner parameter $k \geq 2$.

Definition 3.1. For $0 \leq i \leq 2$, we define the P_i -principal series representation $\pi_{(\nu_i, \sigma_i)}$ of G by

$$\pi_{(\nu_i, \sigma_i)} = \text{Ind}_{P_i}^G(1_{N_i} \otimes e^{\nu_i + \rho_i} \otimes \sigma_i),$$

i.e. $\pi_{(\nu_i, \sigma_i)}$ is the right regular representation of G on the space $H_{(\nu_i, \sigma_i)}$ which is the completion of

$$H_{(\nu_i, \sigma_i)}^\infty = \left\{ f: G \rightarrow V_{\sigma_i} \text{ smooth} \mid \begin{array}{l} f(namx) = e^{\nu_i + \rho_i}(a) \sigma_i(m) f(x) \\ \text{for } n \in N_i, a \in A_i, m \in M_i, x \in G \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_K \|f(k)\|_{\sigma_i}^2 dk.$$

Here V_{σ_i} is a representation space of σ_i and $\|\cdot\|_{\sigma_i}$ is its norm.

4. REPRESENTATIONS OF $K = SO(3)$

4.1. The spinor covering. To describe the finite dimensional representations of $SO(3)$, the simplest way seems to be the one utilizing the double covering $\varphi: SU(2) = Spin(3) \rightarrow SO(3)$. We use the following realization of the double covering φ , which is introduced in [2].

The Hamilton quaternion algebra \mathbf{H} is realized in $M_2(\mathbf{C})$ by

$$\mathbf{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) \mid a, b \in \mathbf{C} \right\}.$$

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e. $SU(2) = \{x \in \mathbf{H} \mid \det x = 1\}$. Let $\mathbf{P} = \{x \in \mathbf{H} \mid \text{tr } x = 0\}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map $\mathbf{P} \ni p \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$ preserve the Euclidean norm $p \mapsto \det p$ and the orientation, hence we have the homomorphism

$$\varphi: SU(2) \rightarrow SO(\mathbf{P}, \det) \simeq SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by $\{\pm 1_2\}$. An explicit expression of the covering map φ is given by

$$\varphi(x) = \begin{pmatrix} p^2 + q^2 - r^2 - s^2 & -2(ps - qr) & 2(pr + qs) \\ 2(ps + qr) & p^2 - q^2 + r^2 - s^2 & -2(pq - rs) \\ -2(pr - qs) & 2(pq + rs) & p^2 - q^2 - r^2 + s^2 \end{pmatrix}$$

for $x = \begin{pmatrix} p + \sqrt{-1}q & r + \sqrt{-1}s \\ -r + \sqrt{-1}s & p - \sqrt{-1}q \end{pmatrix} \in SU(2)$ ($p, q, r, s \in \mathbf{R}$).

By the derivation $d\varphi: \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ of φ , the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to $-2K_{23}$, $2K_{13}$, $-2K_{12}$ with

$$K_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad K_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

respectively.

4.2. Representations of $SU(2)$. The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor product τ_l ($l \in \mathbf{Z}_{\geq 0}$) of the representation $SU(2) \ni g \mapsto (v \mapsto g \cdot v) \in GL(\mathbf{C}^2)$. We use the following realizations of those which are introduced in [2].

Let V_l be the subspace consisting of degree l homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) = f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $\mathfrak{su}(2)$, the derivation of τ_l , denoted by same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k} \mid 0 \leq k \leq l\}$ and the standard generators $\{u_1, u_2, u_3\}$. Namely we have

$$\tau_l(H)v_k = (l - 2k)v_k, \quad \tau_l(E)v_k = -kv_{k-1}, \quad \tau_l(F)v_k = (k - l)v_{k+1}.$$

Here $\{E, H, F\}$ is \mathfrak{sl}_2 -triple defined by

$$H = -\sqrt{-1}u_1, \quad E = \frac{1}{2}(u_2 - \sqrt{-1}u_3), \quad F = -\frac{1}{2}(u_2 + \sqrt{-1}u_3) \in \mathfrak{su}(2)_{\mathbf{C}} = \mathfrak{sl}(2, \mathbf{C}).$$

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect to $\varphi: SU(2) \rightarrow SO(3)$ is that $\tau_l(-1_2) = (-1)^l = 1$, i.e. l is even. For $l \in \mathbf{Z}_{\geq 0}$, we denote the irreducible representation of $SO(3)$ induced from (τ_{2l}, V_{2l}) again by (τ_{2l}, V_{2l}) .

4.3. The adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$. It is known that $\mathfrak{p}_{\mathbf{C}}$ becomes a K -module via the adjoint action of K . Concerning this, we have the following lemma.

Lemma 4.1. *Let $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_4, V_4) and $\{X_j \mid 0 \leq j \leq 4\}$ be a basis of $\mathfrak{p}_{\mathbf{C}}$ defined as follows:*

$$\begin{aligned} X_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\sqrt{-1} \\ 0 & -\sqrt{-1} & -1 \end{pmatrix}, & X_1 &= -\frac{1}{2} \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ \sqrt{-1} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ X_2 &= -\frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & X_3 &= -\frac{1}{2} \begin{pmatrix} 0 & \sqrt{-1} & -1 \\ \sqrt{-1} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ X_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{-1} \\ 0 & \sqrt{-1} & -1 \end{pmatrix}. \end{aligned}$$

Then via the unique isomorphism V_4 and $\mathfrak{p}_{\mathbf{C}}$ as K -modules we have the identification $w_j = X_j$ ($0 \leq j \leq 4$).

Proof. By direct computation, we have the following table of the adjoint actions of the basis $\{d\varphi(E), d\varphi(H), d\varphi(F)\}$ of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$.

	X_0	X_1	X_2	X_3	X_4
$d\varphi(H)$	$4X_0$	$2X_1$	0	$-2X_3$	$-4X_4$
$d\varphi(E)$	0	$-X_0$	$-2X_1$	$-3X_2$	$-4X_3$
$d\varphi(F)$	$-4X_1$	$-3X_2$	$-2X_3$	$-1X_4$	0

TABLE. The adjoint actions of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$.

Comparing the actions in the above table with the actions in Subsection 4.2, we have the assertion. \square

4.4. Clebsch-Gordan coefficients for the representations of $\mathfrak{sl}(2, \mathbf{C})$ with respect to standard basis. In the later sections, we need irreducible decomposition of the tensor product $V \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$ as K -modules for each K -type (τ, V) of $\pi_{(\nu_i, \sigma_i)}$. From the previous arguments, it suffices to consider the irreducible decomposition of $V_l \otimes_{\mathbf{C}} V_4$ as $\mathfrak{sl}(2, \mathbf{C}) = \mathfrak{su}(2)_{\mathbf{C}}$ -modules for arbitrary non-negative integer l .

Generically, the tensor product $V_l \otimes_{\mathbf{C}} V_4$ has five irreducible components V_{l+4} , V_{l+2} , V_l , V_{l-2} and V_{l-4} . Here some components may vanish. We give an explicit expression of a nonzero $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism from each irreducible component to $V_l \otimes_{\mathbf{C}} V_4$ as follows.

Proposition 4.2. *Let $\{v_k^{(l)} \mid 0 \leq k \leq l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\geq 0}$. We put $v_k^{(l)} = 0$ when $k < 0$ or $k > l$.*

If V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish, then we define linear maps $I_{2m}^l: V_{l+2m} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ ($-2 \leq m \leq 2$) by

$$I_{2m}^l(v_k^{(l+2m)}) = \sum_{i=0}^4 A_{[l, 2m; k, i]} \cdot v_{k+2-m-i}^{(l)} \otimes w_i.$$

Here the coefficients $A_{[l, 2m; k, i]} = a(l, 2m; k, i)/d(l, 2m)$ are defined by following formulae.

Formula 1: The coefficients of $I_4^l: V_{l+4} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 4; k, 0) &= (l+4-k)(l+3-k)(l+2-k)(l+1-k), \\ a(l, 4; k, 1) &= 4(l+4-k)(l+3-k)(l+2-k)k, \\ a(l, 4; k, 2) &= 6(l+4-k)(l+3-k)k(k-1), \\ a(l, 4; k, 3) &= 4(l+4-k)k(k-1)(k-2), \\ a(l, 4; k, 4) &= k(k-1)(k-2)(k-3), \\ d(l, 4) &= (l+4)(l+3)(l+2)(l+1). \end{aligned}$$

Formula 2: The coefficients of $I_2^l: V_{l+2} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 2; k, 0) &= (l+2-k)(l+1-k)(l-k), & a(l, 2; k, 1) &= -(l+2-k)(l+1-k)(l-4k), \\ a(l, 2; k, 2) &= -3(l+2-k)(l-2k+2)k, & a(l, 2; k, 3) &= -(3l-4k+8)k(k-1), \\ a(l, 2; k, 4) &= -k(k-1)(k-2), & d(l, 2) &= (l+2)(l+1)l. \end{aligned}$$

Formula 3: The coefficients of $I_0^l: V_l \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 0; k, 0) &= (l-k)(l-1-k), & a(l, 0; k, 1) &= -2(l-k)(l-2k-1), \\ a(l, 0; k, 2) &= (l^2-6kl+6k^2-l), & a(l, 0; k, 3) &= 2(l-2k+1)k, \\ a(l, 0; k, 4) &= k(k-1), & d(l, 0) &= l(l-1). \end{aligned}$$

Formula 4: The coefficients of $I_{-2}^l: V_{l-2} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, -2; k, 0) &= (l-k-2), & a(l, -2; k, 1) &= -(3l-4k-6), & a(l, -2; k, 2) &= 3(l-2k-2), \\ a(l, -2; k, 3) &= -(l-4k-2), & a(l, -2; k, 4) &= -k, & d(l, -2) &= l-2. \end{aligned}$$

Formula 5: The coefficients of $I_{-4}^l: V_{l-4} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, -4; k, 0) &= 1, & a(l, -4; k, 1) &= -4, & a(l, -4; k, 2) &= 6, \\ a(l, -4; k, 3) &= -4, & a(l, -4; k, 4) &= 1, & d(l, -4) &= 1. \end{aligned}$$

Then I_{2m}^l is a generator of $\text{Hom}_{\mathfrak{sl}(2, \mathbf{C})}(V_{l+2m}, V_l \otimes_{\mathbf{C}} V_4)$, which is unique up to scalar multiple.

Proof. We have

$$\begin{aligned} & (\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)}) \\ &= \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot (\tau_l(E)v_{2-m-i}^{(l)}) \otimes w_i + \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot v_{2-m-i}^{(l)} \otimes (\tau_4(E)w_i) \\ &= \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot (-(2-m-i)v_{1-m-i}^{(l)}) \otimes w_i + \sum_{i=1}^4 A_{[l, 2m; 0, i]} \cdot v_{2-m-i}^{(l)} \otimes (-iw_{i-1}) \\ &= -\sum_{i=0}^4 ((2-m-i)A_{[l, 2m; 0, i]} + (i+1)A_{[l, 2m; 0, i+1]}) \cdot v_{1-m-i}^{(l)} \otimes w_i. \end{aligned}$$

Here we put $A_{[l, 2m; 0, 5]} = 0$. By direct computation, we confirm

$$(2-m-i)A_{[l, 2m; 0, i]} + (i+1)A_{[l, 2m; 0, i+1]} = 0$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Hence

$$(\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)}) = 0.$$

Moreover, we have

$$(\tau_l \otimes \tau_4)(H) \circ I_{2m}^l(v_0^{(l+2m)}) = (l+2m)I_{2m}^l(v_0^{(l+2m)}),$$

since

$$\begin{aligned} (\tau_l \otimes \tau_4)(H)(v_i^{(l)} \otimes w_j) &= (\tau_l(H)v_i^{(l)}) \otimes w_j + v_i^{(l)} \otimes (\tau_4(H)w_j) \\ &= (l+4-2i-2j)v_i^{(l)} \otimes w_j. \end{aligned}$$

This means $I_{2m}^l(v_0^{(l+2m)})$ is the highest weight vector of V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ with respect to a Borel subalgebra $(\mathbf{C} \cdot H) \oplus (\mathbf{C} \cdot E)$ of $\mathfrak{sl}(2, \mathbf{C})$.

Therefore, in order to complete the proof, it suffices to confirm

$$(\tau_l \otimes \tau_4)(F) \circ I_{2m}^l(v_k^{(l+2m)}) = I_{2m}^l \circ \tau_{l+2m}(F)(v_k^{(l+2m)})$$

for each $0 \leq k \leq l+2m$.

We confirm these equations by direct computation. \square

The coefficients $A_{[l, 2m; k, i]}$ in the above proposition have the following relations.

Lemma 4.3. *The coefficients $A_{[l, 2m; k, i]}$ in Proposition 4.2 satisfy following relations:*

$$\begin{aligned} A_{[l, 2m; l+2m-k, 0]} &= (-1)^m A_{[l, 2m; k, 4]}, \quad A_{[l, 2m; l+2m-k, 2]} = (-1)^m A_{[l, 2m; k, 2]}, \\ 3\{(k-m+1)A_{[l, 2m; k, 1]} + (l-k+m+1)A_{[l, 2m; k, 3]}\} &= (ml+m^2+m-6)A_{[l, 2m; k, 2]}. \end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq k \leq l+2m$.

Proof. These are obtained by direct computation. \square

4.5. The dual representation of (τ_l, V_l) . We denote by (τ^*, V^*) the dual representation of (τ, V) . Here we note that V_l^* is equivalent to V_l as $SU(2)$ -modules, since irreducible $l+1$ -dimensional representation of $SU(2)$ is unique up to isomorphism.

Lemma 4.4. *Let $\{v_k^{(l)*} \mid 0 \leq k \leq l\}$ is the dual basis of the standard basis $\{v_k^{(l)} \mid 0 \leq k \leq l\}$. Via the unique isomorphism between V_l and V_l^* as K -modules we have the identification*

$$v_k^{(l)} = (-1)^k \frac{(l-k)!k!}{l!} v_{l-k}^{(l)*}$$

for $0 \leq k \leq l$.

Proof. We denote by \langle, \rangle the canonical pairing on $V_l^* \otimes_{\mathbf{C}} V_l$.

Since

$$\langle \tau_l^*(H)v_k^{(l)*}, v_m^{(l)} \rangle = -\langle v_k^{(l)*}, \tau_l(H)v_m^{(l)} \rangle = (2m-l)\delta_{km} = (2k-l)\delta_{km},$$

we have $\tau_l^*(H)v_k^{(l)*} = (2k-l)v_k^{(l)*}$. Similarly, we obtain

$$\tau_l^*(E)v_k^{(l)*} = (k+1)v_{k+1}^{(l)*}, \quad \tau_l^*(F)v_k^{(l)*} = (l-k+1)v_{k-1}^{(l)*}.$$

From these equations, we obtain the assertion. \square

5. THE (\mathfrak{g}, K) -MODULE STRUCTURES OF PRINCIPAL SERIES REPRESENTATIONS

5.1. Irreducible decomposition of $(\pi_{(\nu_0, \sigma_0)}|_K, H_{(\nu_0, \sigma_0)})$ as K -modules. We set

$$L_{(M_0, \sigma_0)}^2(K) = \{f \in L^2(K) \mid f(mx) = \sigma_0(m)f(x) \text{ for a.e. } m \in M, x \in K\}$$

and give a K -module structure by the right regular action of K . Then the restriction map $r_K: H_{(\nu_0, \sigma_0)} \ni f \mapsto f|_K \in L_{(M_0, \sigma_0)}^2(K)$ is an isomorphism of K -modules.

$L^2(K)$ has a $K \times K$ -bimodule structure by the two sided regular action:

$$((k_1, k_2)f)(x) = f(k_1^{-1}xk_2), \quad x \in K, f \in L^2(K), (k_1, k_2) \in K \times K.$$

Then we define a homomorphism $\Phi_l: V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \rightarrow L^2(K)$ of $K \times K$ -bimodules by

$$w \otimes v \mapsto (x \mapsto \langle w, \tau_{2l}(x)v \rangle).$$

Then the Peter-Weyl's theorem tells that

$$\widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} \Phi_l} : \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} V_{2l}^* \otimes_{\mathbf{C}} V_{2l}} \rightarrow L^2(K)$$

is an isomorphism as $K \times K$ -bimodules. Here $\widehat{\bigoplus}$ means a Hilbert space direct sum.

Since $L^2_{(M_0, \sigma_0)}(K) \subset L^2(K)$, we have an irreducible decomposition of $L^2_{(M_0, \sigma_0)}(K)$:

$$L^2_{(M_0, \sigma_0)}(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}}.$$

Here $V[\sigma_0]$ means the σ_0 -isotypic component in $(\tau|_{M_0}, V)$ for a K -module (τ, V) . Therefore we obtain an isomorphism

$$r_K^{-1} \circ \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} \Phi_l} : \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}} \rightarrow H_{(\nu_0, \sigma_0)}.$$

Since M_0 is generated by the two elements

$$m_{0,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_{0,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in M_0,$$

we note that $v \in V_{2l}[\sigma_0]$ if and only if

$$\tau_{2l}(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_0, i} v \quad (i = 1, 2)$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi_1^{-1}(m_{0,1}) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \varphi_1^{-1}(m_{0,2}) = \left\{ \pm \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\},$$

we have $\tau_{2l}(m_{0,1})v_k^{(2l)} = (-1)^k v_{2l-k}^{(2l)}$ and $\tau_{2l}(m_{0,2})v_k^{(2l)} = (-1)^{l-k} v_k^{(2l)}$. Hence we have

$$V_{2l}[\sigma_0] = \bigoplus_{k \in Z(\sigma_0; l)} \mathbf{C} \cdot (v_{2l-k}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)} v_k^{(2l)}),$$

where $\varepsilon(\sigma_0; l) \in \{0, 1\}$ such that $\varepsilon(\sigma_0; l) \equiv l - \sigma_1 - \sigma_2 \pmod{2}$ and

$$Z(\sigma_0; l) = \begin{cases} \{k \in \mathbf{Z} \mid 0 \leq k \leq l, k \equiv l - \sigma_{0,2} \pmod{2}\} & \text{if } \varepsilon(\sigma_0; l) = 0, \\ \{k \in \mathbf{Z} \mid 0 \leq k \leq l-1, k \equiv l - \sigma_{0,2} \pmod{2}\} & \text{if } \varepsilon(\sigma_0; l) = 1. \end{cases}$$

By the identification $V_{2l}^* = V_{2l}$ in Lemma 4.4, we note that $\{v_{2l-k}^{(2l)*} + (-1)^{\varepsilon(\sigma_0; l)} v_k^{(2l)*} \mid k \in Z(\sigma_0; l)\}$ is the basis of $V_{2l}^*[\sigma_0]$.

Now we define the elementary function $s(l; p, q) \in H_{(\nu_0, \sigma_0)}$ by

$$s(l; p, q) = r_K^{-1} \circ \Phi_l^{(j)}((v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0; l)} v_p^{(2l)*}) \otimes v_q^{(2l)})$$

for $l \in \mathbf{Z}_{\geq 0}$, $p \in Z(\sigma_0; l)$ and $0 \leq q \leq 2l$.

For each $p \in Z(\sigma_0; l)$, we put $S(l; p)$ a column vector of degree $2l + 1$ whose $q + 1$ -th component is $s(l; p, q)$, i.e. ${}^t(s(l; p, 0), s(l; p, 1), \dots, s(l; p, 2l))$.

Moreover we denote by $\langle S(l; p) \rangle$ the subspace of $H_{(\nu_0, \sigma_0)}$ generated by the functions in the entries of the vector $S(l; p)$, i.e. $\langle S(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot s(l; p, q) \simeq V_{2l}$. Via the unique isomorphism between $\langle S(l; p) \rangle$ and V_{2l} , we identify $\{s(l; p, q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

Proposition 5.1. *As an unitary representation of K , it has an irreducible decomposition:*

$$H_{(\nu_0, \sigma_0)} \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0])} \otimes_{\mathbf{C}} V_{2l}.$$

Then the τ_{2l} -isotypic component of $\pi_{(\nu_0, \sigma_0)}$ is given by

$$\bigoplus_{p \in Z(\sigma_0; l)} \langle S(l; p) \rangle.$$

Corollary 5.2. *Let $d(\sigma_0; l)$ be the dimension of the space $\text{Hom}_K(V_{2l}, H_{(\nu_0, \sigma_0), K})$ of intertwining operators. Then*

$$d(\sigma_0; l) = \begin{cases} (l+2)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is even,} \\ (l-1)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is odd,} \\ l/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is even,} \\ (l+1)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is odd.} \end{cases}$$

5.2. General setting. Let $H_{(\nu_i, \sigma_i), K}$ be the K -finite part of $H_{(\nu_i, \sigma_i)}$. In order to describe the action of \mathfrak{g} or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of \mathfrak{p} or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

For a K -type (τ_{2l}, V_{2l}) of $\pi_{(\nu_i, \sigma_i)}$ and a nonzero K -homomorphism $\eta: V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$, we define a linear map

$$\tilde{\eta}: \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$$

by $X \otimes v \mapsto \pi_{(\nu_i, \sigma_i)}(X)\eta(v)$. Here we denote differential of $\pi_{(\nu_i, \sigma_i)}$ again by $\pi_{(\nu_i, \sigma_i)}$. Then $\tilde{\eta}$ is K -homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K .

Since

$$V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq V_{2l} \otimes_{\mathbf{C}} V_4 \simeq \bigoplus_{-2 \leq m \leq 2} V_{2(l+m)},$$

there are five injective K -homomorphisms

$$I_{2m}^{2l}: V_{2(l+m)} \rightarrow V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}, \quad -2 \leq m \leq 2$$

for general $l \in \mathbf{Z}_{\geq 0}$. Then we define \mathbf{C} -linear maps

$$\Gamma_{l,m}^i: \text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K}) \rightarrow \text{Hom}_K(V_{2(l+m)}, H_{(\nu_i, \sigma_i), K}), \quad -2 \leq m \leq 2$$

by $\eta \mapsto \tilde{\eta} \circ I_{2m}^{2l}$.

Now we settle two purposes of this paper:

- (i): Describe the injective K -homomorphism I_{2m}^{2l} in terms of the standard basis.
- (ii): Determine the matrix representations of the linear homomorphisms $\Gamma_{l,m}^i$ with respect to the induced basis defined in the next subsection.

We have already accomplished the first purpose in Proposition 4.2. We accomplish the second purpose in Theorem 5.5 and 6.5. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_{(\nu_i, \sigma_i)}[\tau_{2l}]$ and $H_{(\nu_i, \sigma_i)}[\tau_{2(l+m)}]$. Here $H_{(\nu_i, \sigma_i)}[\tau]$ is τ -isotypic component of $H_{(\nu_i, \sigma_i)}$.

5.3. The canonical blocks of elementary functions. Let $\eta: V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$ be a non-zero K -homomorphism. Then we identify η with the column vector of degree $2l+1$ whose $q+1$ -th component is $\eta(v_q^{(2l)})$ for $0 \leq q \leq 2l$, i.e. ${}^t(\eta(v_0^{(2l)}), \eta(v_1^{(2l)}), \dots, \eta(v_{2l}^{(2l)}))$.

By this identification, we identify $S(l; p)$ with the injective K -homomorphism

$$V_{2l} \ni v_q^{(2l)} \mapsto s(l; p, q) \in H_{(\nu_0, \sigma_0), K}, \quad 0 \leq q \leq 2l$$

for $p \in Z(\sigma_0; l)$. We note that $\{S(l; p) \mid p \in Z(\sigma_0; l)\}$ is a basis of $\text{Hom}_K(V_{2l}, H_{(\nu_0, \sigma_0), K})$ and we call it *the induced basis from the standard basis*.

We define a certain matrix of elementary functions corresponding to the induced basis $\{S(l; p) \mid p \in Z(\sigma_0; l)\}$ of $\text{Hom}_K(V_{2l}, H_{(\nu_0, \sigma_0), K})$ for each K -type τ_{2l} of our principal series representation $\pi_{(\nu_0, \sigma_0)}$.

Definition 5.3. *The following $(2l+1) \times d(\sigma_0; l)$ matrix $\mathbf{S}(\sigma_0; l)$ is called the canonical block of elementary functions for τ_{2l} -isotypic component: When $(\sigma_{0,1}, \sigma_{0,2}) = (0, 0)$, we consider the matrix*

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l)) & \text{if } l \text{ is even,} \\ (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l-2)) & \text{if } l \text{ is odd.} \end{cases}$$

When $(\sigma_{0,1}, \sigma_{0,2}) = (1, 0)$, we consider the matrix

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l-2)) & \text{if } l \text{ is even,} \\ (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l)) & \text{if } l \text{ is odd.} \end{cases}$$

When $\sigma_{0,2} = 1$, we consider the matrix

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l-1)) & \text{if } l \text{ is even,} \\ (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l-1)) & \text{if } l \text{ is odd.} \end{cases}$$

5.4. The $\mathbf{p}_\mathbf{C}$ -matrix corresponding to I_{2m}^{2l} . For two integers c_0, c_1 such that $c_0 \leq c_1$ and a rational function $f(x)$ in the variable x , we denote by

$$\text{Diag}_{c_0 \leq n \leq c_1} (f(n))$$

the diagonal matrix of size $c_1 - c_0 + 1$ with an entry $f(n)$ at the $(n - c_0 + 1, n - c_0 + 1)$ -th component. Let $\mathbf{e}_i^{(l)}$ ($0 \leq i \leq l$) be the column unit vector of degree $l+1$ with its $i+1$ -th component 1 and the remaining components 0. Moreover, let $\mathbf{e}_i^{(l)}$ be the column zero vector of degree $l+1$ when $i < 0$ or $l < i$.

In this subsection, we define $\mathbf{p}_\mathbf{C}$ -matrix $\mathfrak{C}_{l,m}$ of size $(2(l+m)+1) \times (2l+1)$ corresponding to I_{2m}^{2l} with respect to the standard basis.

Let $\sum_{i=0}^4 \iota_i^{(l,m)} \otimes X_i$ be the image of I_{2m}^{2l} by the composite of natural linear maps

$$\text{Hom}_K(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathbf{p}_\mathbf{C}) \rightarrow \text{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathbf{p}_\mathbf{C}) \simeq \text{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l}) \otimes_{\mathbf{C}} \mathbf{p}_\mathbf{C}.$$

Then we define $\mathbf{p}_\mathbf{C}$ -matrix $\mathfrak{C}_{l,m} = \sum_{i=0}^4 R(\iota_i^{(l,m)}) \otimes X_i$ where $R(\iota_i^{(l,m)})$ is the matrix representation of $\iota_i^{(l,m)}$ with respect to the standard basis. Explicit expression of the matrix $R(\iota_i^{(l,m)})$ of size $(2(l+m)+1) \times (2l+1)$ is given by

$$\begin{aligned} & \left(O_{2(l+m)+1, m+2}, R(\iota_0^{(l,m)}), O_{2(l+m)+1, m+2} \right) \\ &= \left(O_{2(l+m)+1, 4-i}, \text{Diag}_{0 \leq k \leq 2(l+m)} (A_{[2l, 2m; k, i]}), O_{2(l+m)+1, i} \right) \end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Here we erase the symbol $O_{m,n}$ when $m = 0$ or $n = 0$.

For a column vector $\mathbf{v} = {}^t(v_0, v_1, \dots, v_{2l}) \in (H_{(\nu_i, \sigma_i), K})^{\oplus 2l+1}$ which is identified with an element of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$, we define $\mathfrak{C}_{l,m} \mathbf{v} \in (H_{(\nu_i, \sigma_i), K})^{\oplus 2(l+m)+1} \simeq \mathbf{C}^{2(l+m)+1} \otimes_{\mathbf{C}} H_{(\nu_0, \sigma_0), K}$ by

$$\mathfrak{C}_{l,m} \mathbf{v} = \sum_{\substack{0 \leq i \leq 4 \\ 0 \leq q \leq 2l}} (R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes (\pi_{(\nu_i, \sigma_i)}(X_i) v_q).$$

Here $R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}$ is the ordinal product of matrices $R(\iota_i^{(l,m)})$ and $\mathbf{e}_q^{(2l)}$.

From the definition of $\mathfrak{C}_{l,m}$, we note that the vector $\mathfrak{C}_{l,m} \mathbf{v}$ is identified with the image of \mathbf{v} by $\Gamma_{l,m}^i$.

5.5. The contiguous relations.

Lemma 5.4. *The standard basis X_i ($0 \leq i \leq 4$) in $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned} X_0 &= -2\sqrt{-1}E_{e_2-e_3} + H_2 + \sqrt{-1}K_{23}, \\ X_1 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\ X_2 &= -\frac{1}{3}(2H_1 - H_2), \\ X_3 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \\ X_4 &= 2\sqrt{-1}E_{e_2-e_3} + H_2 - \sqrt{-1}K_{23}. \end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 4.1. \square

We give the matrix representation of $\Gamma_{l,m}^0$ with respect to the induced basis as follows.

Theorem 5.5. *For $l \in \mathbf{Z}_{\geq 0}$, $-2 \leq m \leq 2$ such that $d(\sigma_0; l) > 0$ and $d(\sigma_0; l+m) > 0$, we have*

$$(5.1) \quad \mathfrak{C}_{l,m} \mathbf{S}(\sigma_0; l) = \mathbf{S}(\sigma_0; l+m) \cdot R(\Gamma_{l,m}^0)$$

with the matrix representation $R(\Gamma_{l,m}^0) \in M_{d(\sigma_0; l+m), d(\sigma_0; l)}(\mathbf{C})$ of $\Gamma_{l,m}^0$ with respect to the induced basis $\{S(l; p) \mid p \in Z(\sigma_0; l)\}$:

Explicit expressions of the matrix $R(\Gamma_{l,m}^0)$ of size $d(\sigma_0; l+m) \times d(\sigma_0; l)$ is given as follows:

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (2\mathbf{Z})$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$\begin{aligned} \begin{pmatrix} O_{n(\sigma_0; l, m), d(\sigma_0; l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix} &= \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), -1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-1} \\ O_{1, d(\sigma_0; l)} \end{pmatrix} + \begin{pmatrix} O_{1, d(\sigma_0; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), 0]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-1} \end{pmatrix} \\ &+ \begin{pmatrix} O_{2, d(\sigma_0; l)-1} & O_{2, 1} \\ \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), 1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-2} & \gamma_{[l, m; l, 1]}^{(0)} \cdot e_{d(\sigma_0; l)-3}^{(d(\sigma_0; l)-2)} \end{pmatrix}. \end{aligned}$$

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (1 + 2\mathbf{Z})$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$\begin{aligned} \begin{pmatrix} O_{n(\sigma_0; l, m), d(\sigma_0; l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix} &= \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), -1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-1} \\ O_{1, d(\sigma_0; l)} \end{pmatrix} + \begin{pmatrix} O_{1, d(\sigma_0; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), 0]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-1} \end{pmatrix} \\ &+ \begin{pmatrix} O_{2, d(\sigma_0; l)-1} & O_{2, 1} \\ \text{Diag} \left(\gamma_{[l, m; 2k+\delta(\sigma_0; l), 1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l)-2} & O_{d(\sigma_0; l)-1, 1} \end{pmatrix}. \end{aligned}$$

When $\sigma_{0,2} = 0$, $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (2\mathbf{Z})$ and $d(\sigma_0; l) = 1$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$R(\Gamma_{l,m}^0) = \left(\gamma_{[l, m; \delta(\sigma_0; l), -1]}^{(0)} \right).$$

When $\sigma_{0,2} = 0$, $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (2\mathbf{Z})$ and $d(\sigma_0; l) > 1$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$\begin{pmatrix} O_{n(\sigma_0; l, m), d(\sigma_0; l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix} = \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), -1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \\ \begin{pmatrix} O_{1, d(\sigma_0; l) - 1} & 0 \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 0]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 2} & O_{d(\sigma_0; l) - 1, 1} \end{pmatrix} \\ \begin{pmatrix} O_{2, d(\sigma_0; l) - 2} & O_{2, 1} & O_{2, 1} \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 3} & O_{d(\sigma_0; l) - 2, 1} & -\gamma_{[l, m; l, 1]}^{(0)} \cdot e_{d(\sigma_0; l) - 3}^{(d(\sigma_0; l) - 3)} \end{pmatrix} \end{pmatrix}.$$

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (1 + 2\mathbf{Z})$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$\begin{pmatrix} O_{n(\sigma_0; l, m), d(\sigma_0; l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix} = \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), -1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \\ O_{2, d(\sigma_0; l)} \end{pmatrix} + \begin{pmatrix} O_{1, d(\sigma_0; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 0]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \\ O_{1, d(\sigma_0; l)} \end{pmatrix} + \begin{pmatrix} O_{2, d(\sigma_0; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \end{pmatrix}.$$

When $\sigma_{0,2} = 1$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$\begin{pmatrix} O_{n(\sigma_0; l, m), d(\sigma_0; l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix} = \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), -1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \\ O_{1, d(\sigma_0; l)} \end{pmatrix} + \begin{pmatrix} O_{1, d(\sigma_0; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 0]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 1} \end{pmatrix} + \begin{pmatrix} O_{2, d(\sigma_0; l) - 1} & O_{2, 1} \\ \text{Diag} \left(\gamma_{[l, m; 2k + \delta(\sigma_0; l), 1]}^{(0)} \right)_{0 \leq k \leq d(\sigma_0; l) - 2} & (-1)^{\varepsilon(\sigma_0; l + m)} \gamma_{[l, m; l - 1, 1]}^{(0)} \cdot e_{d(\sigma_0; l) - 2}^{(d(\sigma_0; l) - 2)} \end{pmatrix}.$$

Here

$$\begin{aligned} \gamma_{[l, m; p, 1]}^{(0)} &= (\nu_{0,2} + \rho_{0,2} - l + p) A_{[2l, 2m; 2l - p + m - 2, 0]}, \\ \gamma_{[l, m; p, 0]}^{(0)} &= -\frac{1}{3} \left(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2} + lm - 3 + \frac{m(m+1)}{2} \right) A_{[2l, 2m; 2l - p + m, 2]}, \\ \gamma_{[l, m; p, -1]}^{(0)} &= (\nu_{0,2} + \rho_{0,2} + l - p) A_{[2l, 2m; 2l - p + m + 2, 4]}, \\ n(\sigma_0; l, m) &= \begin{cases} (2 - m)/2 & \text{if } m \in \{0, \pm 2\}, \\ (3 - m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (2\mathbf{Z}), \\ (1 - m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (1 + 2\mathbf{Z}), \end{cases} \end{aligned}$$

and $\delta(\sigma_0; l) \in \{0, 1\}$ such that $\delta(\sigma_0; l) \equiv l - \sigma_{0,2} \pmod{2}$.

In the above equations, we put $A_{[2l, 2m; k, i]} = 0$ for $k < 0$ or $k > 2(l + m)$, and erase the symbols $\text{Diag} (f(n))$, $O_{0,n}$, $O_{m,0}$ and $\mathbf{e}_i^{(-1)}$.

Proof. Since

$$s(l; p, q)(1_3) = \langle (v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0; l)} v_p^{(2l)*}), v_q^{(2l)} \rangle = \delta_{2l-p, q} + (-1)^{\varepsilon(\sigma_0; l)} \delta_{pq},$$

we have

$$(5.2) \quad S(l; p)(1_3) = \mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)} \mathbf{e}_p^{(2l)}.$$

Hence $S(l; p)(1_3)$ ($p \in Z(\sigma_0; l)$) are linearly independent over \mathbf{C} . Thus we note that it suffices to evaluate the both side of the equation (5.1) at $1_3 \in G$.

First, we compute $\{\pi_{(\nu_0, \sigma_0)}(X_i)s(l; p, q)\}(1_3)$ for $0 \leq i \leq 4$, $p \in Z(\sigma_0; l)$ and $0 \leq q \leq 2l$. Since $\{s(l; p, q) \mid 0 \leq q \leq 2l\}$ is the standard basis of $\langle S(l; p) \rangle$, we obtain

$$\begin{aligned} \{\pi_{(\nu_0, \sigma_0)}(\sqrt{-1}K_{23})s(l; p, q)\}(1_3) &= (l - q)s(l; p, q)(1_3) \\ &= (l - q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{pq}), \\ \{\pi_{(\nu_0, \sigma_0)}(K_{13} + \sqrt{-1}K_{12})s(l; p, q)\}(1_3) &= -q \cdot s(l; p, q - 1)(1_3) \\ &= -q(\delta_{2l-p+1q} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{p+1q}), \\ \{\pi_{(\nu_0, \sigma_0)}(K_{13} - \sqrt{-1}K_{12})s(l; p, q)\}(1_3) &= (2l - q)s(l; p, q + 1)(1_3) \\ &= (2l - q)(\delta_{2l-p-1q} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{p-1q}). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \{\pi_{(\nu_0, \sigma_0)}(E_\alpha)s(l; p, q)\}(1_3) &= 0, & \alpha \in \Sigma^+, \\ \{\pi_{(\nu_0, \sigma_0)}(H_i)s(l; p, q)\}(1_3) &= (\nu_{0,i} + \rho_{0,i})s(l; p, q)(1_3) \\ &= (\nu_{0,i} + \rho_{0,i})(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{pq}), & i = 1, 2, \end{aligned}$$

from the definition of principal series representation. From these computations and Iwasawa decomposition in Lemma 5.4, we obtain

$$\begin{aligned} \{\pi_{(\nu_0, \sigma_0)}(X_0)s(l; p, q)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} + l - q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{pq}), \\ \{\pi_{(\nu_0, \sigma_0)}(X_1)s(l; p, q)\}(1_3) &= -\frac{q}{2}(\delta_{2l-p+1q} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{p+1q}), \\ \{\pi_{(\nu_0, \sigma_0)}(X_2)s(l; p, q)\}(1_3) &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{pq}), \\ \{\pi_{(\nu_0, \sigma_0)}(X_3)s(l; p, q)\}(1_3) &= -\frac{2l - q}{2}(\delta_{2l-p-1q} - (-1)^{\varepsilon(\sigma_0; l)}\delta_{p-1q}), \\ \{\pi_{(\nu_0, \sigma_0)}(X_4)s(l; p, q)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} - l + q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0; l)}\delta_{pq}). \end{aligned}$$

We set

$$\pi_{(\nu_0, \sigma_0)}(X_i)S(l; p) = \sum_{0 \leq q \leq 2l} \mathbf{e}_q^{(2l)} \otimes (\pi_{(\nu_0, \sigma_0)}(X_i)s(l; p, q)).$$

Then we obtain

$$\begin{aligned} \{\pi_{(\nu_0, \sigma_0)}(X_0)S(l; p)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_p^{(2l)}, \\ \{\pi_{(\nu_0, \sigma_0)}(X_1)S(l; p)\}(1_3) &= -\frac{2l - p + 1}{2}\mathbf{e}_{2l-p+1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{p + 1}{2}\mathbf{e}_{p+1}^{(2l)}, \\ \{\pi_{(\nu_0, \sigma_0)}(X_2)S(l; p)\}(1_3) &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}\mathbf{e}_p^{(2l)}), \\ \{\pi_{(\nu_0, \sigma_0)}(X_3)S(l; p)\}(1_3) &= -\frac{p + 1}{2}\mathbf{e}_{2l-p-1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{2l - p + 1}{2}\mathbf{e}_{p-1}^{(2l)}, \\ \{\pi_{(\nu_0, \sigma_0)}(X_4)S(l; p)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_p^{(2l)}. \end{aligned}$$

Let us compute $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$. By the above equations, we have

$$\begin{aligned}
\{\mathfrak{C}_{l,m}S(l;p)\}(1_3) &= \sum_{\substack{0 \leq i \leq 4 \\ 0 \leq q \leq 2l}} (R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes \{(\pi_{(\nu_0, \sigma_0)}(X_i)s(l;p,q))\}(1_3) \\
&= \sum_{0 \leq i \leq 4} R(\iota_i^{(l,m)}) \cdot \{(\pi_{(\nu_0, \sigma_0)}(X_i)S(l;p))\}(1_3) \\
&= R(\iota_0^{(l,m)}) \cdot \{(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_p^{(2l)}\} \\
&\quad + R(\iota_1^{(l,m)}) \cdot \left\{ -\frac{2l-p+1}{2}\mathbf{e}_{2l-p+1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{p+1}{2}\mathbf{e}_{p+1}^{(2l)} \right\} \\
&\quad + R(\iota_2^{(l,m)}) \cdot \left\{ -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}\mathbf{e}_p^{(2l)}) \right\} \\
&\quad + R(\iota_3^{(l,m)}) \cdot \left\{ -\frac{p+1}{2}\mathbf{e}_{2l-p-1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{2l-p+1}{2}\mathbf{e}_{p-1}^{(2l)} \right\} \\
&\quad + R(\iota_4^{(l,m)}) \cdot \{(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_p^{(2l)}\}.
\end{aligned}$$

Since

$$R(\iota_i^{(l,m)})\mathbf{e}_q^{(2l)} = A_{[2l, 2m; i+q+m-2, i]}\mathbf{e}_{i+q+m-2}^{(2(l+m))}, \quad -2 \leq m \leq 2,$$

we obtain

$$(5.3) \quad \{\mathfrak{C}_{l,m}S(l;p)\}(1_3) = \sum_{-1 \leq i \leq 1} \{\alpha_{[l, m; p, i]}\mathbf{e}_{2(l+m)-(p+m+2i)}^{(2(l+m))} + (-1)^{\varepsilon(\sigma_0; l)}\beta_{[l, m; p, i]}\mathbf{e}_{p+m+2i}^{(2(l+m))}\},$$

where

$$\begin{aligned}
\alpha_{[l, m; p, 1]} &= (\nu_{0,2} + \rho_{0,2} - l + p)A_{[2l, 2m; 2l-p+m-2, 0]}, \\
\alpha_{[l, m; p, 0]} &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{[2l, 2m; 2l-p+m, 2]} \\
&\quad - \frac{2l-p+1}{2}A_{[2l, 2m; 2l-p+m, 1]} - \frac{p+1}{2}A_{[2l, 2m; 2l-p+m, 3]}, \\
\alpha_{[l, m; p, -1]} &= (\nu_{0,2} + \rho_{0,2} + l - p)A_{[2l, 2m; 2l-p+m+2, 4]}, \\
\beta_{[l, m; p, 1]} &= (\nu_{0,2} + \rho_{0,2} - l + p)A_{[2l, 2m; p+m+2, 4]}, \\
\beta_{[l, m; p, 0]} &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{[2l, 2m; p+m, 2]} \\
&\quad - \frac{p+1}{2}A_{[2l, 2m; p+m, 1]} - \frac{2l-p+1}{2}A_{[2l, 2m; p+m, 3]}, \\
\beta_{[l, m; p, -1]} &= (\nu_{0,2} + \rho_{0,2} + l - p)A_{[2l, 2m; p+m-2, 0]}.
\end{aligned}$$

By the relations of the coefficients $A_{[2l, 2m; k, i]}$ in Lemma 4.3, we see that

$$\alpha_{[l, m; p, i]} = (-1)^m \beta_{[l, m; p, i]} = \gamma_{[l, m; p, i]}^{(0)}, \quad -1 \leq i \leq 1.$$

Therefore, (5.3) become

$$(5.4) \quad \{\mathfrak{C}_{l,m}S(l;p)\}(1_3) = \sum_{-1 \leq i \leq 1} \gamma_{[l, m; p, i]}^{(0)} \{\mathbf{e}_{2(l+m)-(p+m+2i)}^{(2(l+m))} + (-1)^{\varepsilon(\sigma_0; l)+m}\mathbf{e}_{p+m+2i}^{(2(l+m))}\}.$$

From the equations (5.2), (5.4) and

$$\varepsilon(\sigma_0; l) + m \equiv \varepsilon(\sigma_0; l+m) \pmod{2},$$

we obtain the assertion. \square

6. THE (\mathfrak{g}, K) -MODULE STRUCTURES OF THE GENERALIZED PRINCIPAL SERIES REPRESENTATIONS

In this section, we set $i = 1$ or 2 .

6.1. Discrete series representations of $SL^\pm(2, \mathbf{R})$. We set $y_0 = \text{diag}(1, -1) \in O(2)$. Then a discrete series representation (D_k, V_{D_k}) of $SL^\pm(2, \mathbf{R})$ is uniquely determined by specifying the $G' = SL(2, \mathbf{R})$ -module structure together with the action of y_0 . Since $D_k|_{G'} = D_k^+ \oplus D_k^-$ and $D_k^+ \oplus D_k^-$ is identified with G' -submodule of the principal series representation $(\pi_{(\nu, \sigma)}, H_{(\nu, \sigma)})$ of G' by Proposition 2.2, we obtain the following realization of (D_k, V_{D_k}) :

$$V_{D_k, O(2)} = \bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} W_{k+2\alpha} \quad (W_p = \mathbf{C} \cdot \chi_p + \mathbf{C} \cdot \chi_{-p})$$

and

$$\begin{aligned} D_k(w)\chi_p &= \sqrt{-1}p\chi_p, & D_k(x_+)\chi_p &= (k+p)\chi_{p+2}, & D_k(x_-)\chi_p &= (k-p)\chi_{p-2}, \\ D_k(\kappa_t)\chi_p &= e^{\sqrt{-1}pt}\chi_p & (t \in \mathbf{R}), & & D_k(y_0)\chi_p &= \chi_{-p}. \end{aligned}$$

Here we denote differential of D_k again by D_k and the $O(2)$ -finite part of V_{D_k} by $V_{D_k, O(2)}$.

6.2. Irreducible decompositions of $(\pi_{(\nu_1, \sigma_1)}|_K, H_{(\nu_1, \sigma_1)})$ and $(\pi_{(\nu_2, \sigma_2)}|_K, H_{(\nu_2, \sigma_2)})$ as K -modules. We analyze the K -type of the representation space $H_{(\nu_i, \sigma_i)}$ of the P_i -principal series representation. the target space V_{σ_i} of functions \mathbf{f} in $H_{(\nu_i, \sigma_i)}$ has a decomposition:

$$V_{\sigma_i} = V_{D_k} = \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} W_{k+2\alpha}}.$$

Denote the corresponding decomposition of \mathbf{f} by

$$\mathbf{f}(x) = \sum_{\alpha=0}^{\infty} (f_{k+2\alpha}(x) \otimes \chi_{k+2\alpha} + f_{-(k+2\alpha)}(x) \otimes \chi_{-(k+2\alpha)}).$$

From the definition of the space $H_{(\nu_i, \sigma_i)}$, we have

$$\mathbf{f}|_K(mx) = \sigma_i(m)\mathbf{f}|_K(x) \quad (\text{a.e. } x \in K, m \in K_i = M_i \cap K \simeq O(2)).$$

For $m = m_i(\kappa_t)$, $m_i(y_0)$, comparing the coefficients of χ_p in the left hand side with those in the right hand side, we have the equations

$$f_p|_K(m_i(\kappa_t)x) = e^{\sqrt{-1}pt}f_p|_K(x), \quad f_p|_K(m_i(y_0)x) = f_{-p}|_K(x).$$

Moreover, from the equality of inner products

$$\int_K \|\mathbf{f}|_K(x)\|_{\sigma_i}^2 dx = \sum_{\varepsilon \in \{\pm 1\}, \alpha \in \mathbf{Z}_{\geq 0}} \left\{ \int_K |f_{\varepsilon(k+2\alpha)}|_K(x)| dx \right\} \|\chi_{\varepsilon(k+2\alpha)}\|_{\sigma_i}^2,$$

we have $f_p|_K \in L^2(K)$. Therefore $\mathbf{f}|_K$ belongs to

$$\widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} L_i^2(K; W_{k+2\alpha})}$$

where

$$\begin{aligned} L_i^2(K; W_p) &= \{\mathbf{f}: K \rightarrow W_p \mid \mathbf{f}(x) = f(x) \otimes \chi_p + f(m_i(y_0)x) \otimes \chi_{-p}, f \in L_{(K_i^\circ, \chi_p)}^2(K), x \in K\}, \\ L_{(K_i^\circ, \chi_p)}^2(K) &= \{f \in L^2(K) \mid f(m_i(\kappa_t)x) = e^{\sqrt{-1}pt}f(x), m_i(\kappa_t) \in K_i^\circ, x \in K\}. \end{aligned}$$

Here K_i° means the connected component of K_i , which is isomorphic to $SO(2)$. We easily see that the restriction map

$$r_K^{(i)}: H_{(\nu_i, \sigma_i)} \ni \mathbf{f} \mapsto \mathbf{f}|_K \in \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} L_i^2(K; W_{k+2\alpha})}$$

is a K -isomorphism.

By the Peter-Weyl's theorem, we have an irreducible decomposition of $L_{(K_i^\circ, \chi_p)}^2(K)$:

$$L_{(K_i^\circ, \chi_p)}^2(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\xi_{(i; -p)}])} \otimes_{\mathbf{C}} V_{2l}.$$

Here

$$\xi_{(i; p)}: K_i^\circ \ni m_i(\kappa_t) \mapsto e^{\sqrt{-1}pt} \in \mathbf{C}^\times$$

and $V[\xi_{(i; p)}]$ means the $\xi_{(i; p)}$ -isotypic component in $(\tau|_{K_i^\circ}, V)$ for a K -module (τ, V) .

In this section, we denote by $\{v_{1,q}^{(2l)} \mid 0 \leq q \leq 2l\}$ the standard basis of V_{2l} . We define another basis $\{v_{2,q}^{(2l)} \mid 0 \leq q \leq 2l\}$ of V_{2l} by

$$v_{2,q}^{(2l)} = \tau_{2l}(u_c)v_{1,q}^{(2l)} = \frac{1}{2l}(x+y)^q(-x+y)^{2l-q} \quad (0 \leq q \leq 2l)$$

where

$$u_c = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3).$$

We note that $v \in V_{2l}[\xi_{(i; -p)}]$ if and only if

$$\tau_{2l}(m_i(\kappa_t))v = \xi_{(i; -p)}(m_i(\kappa_t))v = e^{-\sqrt{-1}pt}v \quad (t \in \mathbf{R})$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi^{-1}(m_1(\kappa_t)) = \varphi^{-1}(u_c^{-1}m_2(\kappa_t)u_c) = \left\{ \pm \operatorname{diag}(e^{-\sqrt{-1}t/2}, e^{\sqrt{-1}t/2}) \right\},$$

we have $\tau_{2l}(m_i(\kappa_t))v_{i,q}^{(2l)} = e^{\sqrt{-1}(q-l)t}v_{i,q}^{(2l)}$. Hence we have

$$V_{2l}[\xi_{(i; -p)}] = \begin{cases} \mathbf{C} \cdot v_{i, l-p}^{(2l)} & \text{if } -l \leq p \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

By the identification $V_{2l}^* = V_{2l}$ in Lemma 4.4, we obtain

$$L_{(K_i^\circ, \chi_p)}^2(K) \simeq \widehat{\bigoplus_{\substack{l \in \mathbf{Z}_{\geq 0} \\ -l \leq p \leq l}} (\mathbf{C} \cdot v_{i, l+p}^{(2l)*})} \otimes_{\mathbf{C}} V_{2l}.$$

Moreover, since

$$\varphi^{-1}(m_1(y_0)) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \varphi^{-1}(u_c^{-1}m_2(y_0)u_c) = \left\{ \pm \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right\},$$

we have

$$\tau_{2l}^*(m_1(y_0)^{-1})v_{1, l+p}^{(2l)*} = (-1)^{l+p}v_{1, l-p}^{(2l)*}, \quad \tau_{2l}^*(m_2(y_0)^{-1})v_{2, l+p}^{(2l)*} = (-1)^l v_{2, l-p}^{(2l)*}.$$

For $0 \leq p \leq l - k$ such that $p \equiv l - k \pmod{2}$, we define the elementary function $t_i(l; p, q) \in H_{(\nu_i, \sigma_i)}$ by

$$t_i(l; p, q) = r_K^{(i)-1}(\tilde{t}_i(l; p, q))$$

where

$$\begin{aligned}\tilde{t}_1(l; p, q)(x) &= \langle v_{1, 2l-p}^{(2l)*}, \tau_{2l}(x)v_{1, q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^p \langle v_{1, p}^{(2l)*}, \tau_{2l}(x)v_{1, q}^{(2l)} \rangle \otimes \chi_{p-l}, \\ \tilde{t}_2(l; p, q)(x) &= \langle v_{2, 2l-p}^{(2l)*}, \tau_{2l}(x)v_{1, q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^l \langle v_{2, p}^{(2l)*}, \tau_{2l}(x)v_{1, q}^{(2l)} \rangle \otimes \chi_{p-l}.\end{aligned}$$

Let $T_i(l; p)$ be a column vector of degree $2l + 1$ with its $q + 1$ -th component $t_i(l; p, q)$, i.e. ${}^t(t_i(l; p, 0), t_i(l; p, 1), \dots, t_i(l; p, 2l))$.

Moreover we denote by $\langle T_i(l; p) \rangle$ the subspace of $H_{(\nu_i, \sigma_i)}$ generated by the functions in the entries of the vector $T_i(l; p)$, i.e. $\langle T_i(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot t_i(l; p, q) \simeq V_{2l}$. Via the unique isomorphism between $\langle T_i(l; p) \rangle$ and V_{2l} , we identify $\{t_i(l; p, q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

Proposition 6.1. *As an unitary representation of K , it has an irreducible decomposition:*

$$H_{(\nu_i, \sigma_i)} = \widehat{\bigoplus_{\substack{l \in \mathbf{Z}_{\geq 0}, \ 0 \leq p \leq l-k \\ p \equiv l-k \pmod{2}}} \langle T_i(l; p) \rangle}$$

for $i = 1, 2$. Then the τ_{2l} -isotypic component of $\pi_{(\nu_i, \sigma_i)}$ is given by

$$\bigoplus_{\substack{0 \leq p \leq l-k \\ p \equiv l-k \pmod{2}}} \langle T_i(l; p) \rangle.$$

Corollary 6.2. *Let $d(\sigma_i; l)$ be the dimension of the space $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ of intertwining operators. Then*

$$d(\sigma_i; l) = \begin{cases} (l - k + 2)/2 & \text{if } k \leq l \text{ and } l - k \text{ is even,} \\ (l - k + 1)/2 & \text{if } k \leq l \text{ and } l - k \text{ is odd,} \\ 0 & \text{if } k > l. \end{cases}$$

6.3. The canonical blocks of elementary functions. By the identification introduced in Subsection 5.3, we identify $T_i(l; p)$ with the injective K -homomorphism

$$V_{2l} \ni v_{1, q}^{(2l)} \mapsto t_i(l; p, q) \in H_{(\nu_i, \sigma_i), K}, \quad 0 \leq q \leq 2l$$

for $0 \leq p \leq l - k$ such that $p \equiv l - k \pmod{2}$. We note that $\{T_i(l; p) \mid 0 \leq p \leq l - k, p \equiv l - k \pmod{2}\}$ is a basis of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ and we call it *the induced basis from the standard basis*.

We define a certain matrix of elementary functions corresponding to the induced basis $\{T_i(l; p) \mid 0 \leq p \leq l - k, p \equiv l - k \pmod{2}\}$ of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ for each K -type τ_{2l} of our P_i -principal series representation $\pi_{(\nu_i, \sigma_i)}$.

Definition 6.3. *For $l \in \mathbf{Z}_{\geq 0}$ such that $d(\sigma_i; l) > 0$, the following $(2l + 1) \times d(\sigma_i; l)$ matrix $\mathbf{T}_i(\sigma_i; l)$ is called the canonical block of elementary functions for τ_{2l} -isotypic component: When $l - k$ is even, we consider the matrix*

$$\mathbf{T}_i(\sigma_i; l) = (T_i(l; 0), T_i(l; 2), T_i(l; 4), \dots, T_i(l; l - k)).$$

When $l - k$ is odd, we consider the matrix

$$\mathbf{T}_i(\sigma_i; l) = (T_i(l; 1), T_i(l; 3), T_i(l; 5), \dots, T_i(l; l - k)).$$

6.4. The contiguous relations.

Lemma 6.4. (i) The standard basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{1,\mathbf{C}} \oplus \mathfrak{a}_{1,\mathbf{C}} \oplus \mathfrak{m}_{1,\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X_0 &= m_1(x_-), & X_1 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\ X_2 &= -\frac{1}{3}H^{(1)}, & X_3 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), & X_4 &= m_1(x_+). \end{aligned}$$

(ii) The basis $\{X'_j = u_c X_j u_c^{-1} \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{2,\mathbf{C}} \oplus \mathfrak{a}_{2,\mathbf{C}} \oplus \mathfrak{m}_{2,\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X'_0 &= -m_2(x_-), & X'_1 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_2-e_3}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{23}), \\ X'_2 &= \frac{1}{3}H^{(2)}, & X'_3 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_2-e_3}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{23}), & X'_4 &= -m_2(x_+), \end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 4.1. \square

We give the matrix representation of $\Gamma_{l,m}^i$ with respect to the induced basis as follows.

Theorem 6.5. For $i = 1, 2$ and $-2 \leq m \leq 2$, we have an following equation with the matrix representation $R(\Gamma_{l,m}^i) \in M_{d(\sigma_i; l+m), d(\sigma_i; l)}(\mathbf{C})$ of $\Gamma_{l,m}^i$ with respect to the induced basis $\{T_i(l; p) \mid 0 \leq p \leq l-k, p \equiv l-k \pmod{2}\}$:

$$(6.1) \quad \mathfrak{C}_{l,m} \mathbf{T}_i(\sigma_i; l) = \mathbf{T}_i(\sigma_i; l+m) \cdot R(\Gamma_{l,m}^i).$$

Explicit expressions of the matrix $R(\Gamma_{l,m}^i)$ of size $d(\sigma_i; l+m) \times d(\sigma_i; l)$ is given as follows:

The matrix $R(\Gamma_{l,m}^i)$ is given by

$$\begin{aligned} \begin{pmatrix} O_{n(\sigma_i; l, m), d(\sigma_i; l)} \\ R(\Gamma_{l,m}^i) \end{pmatrix} &= \begin{pmatrix} \text{Diag} \left(\gamma_{[l, m; 2j+\delta(\sigma_i; l), -1]}^{(i)} \right)_{0 \leq j \leq d(\sigma_i; l)-1} \\ O_{1, d(\sigma_i; l)} \end{pmatrix} + \begin{pmatrix} O_{1, d(\sigma_i; l)} \\ \text{Diag} \left(\gamma_{[l, m; 2j+\delta(\sigma_i; l), 0]}^{(i)} \right)_{0 \leq j \leq d(\sigma_i; l)-1} \end{pmatrix} \\ &+ \begin{pmatrix} O_{2, d(\sigma_i; l)-1} & O_{2, 1} \\ \text{Diag} \left(\gamma_{[l, m; 2j+\delta(\sigma_i; l), 1]}^{(i)} \right)_{0 \leq j \leq d(\sigma_i; l)-2} & O_{d(\sigma_i; l)-1, 1} \end{pmatrix}. \end{aligned}$$

Here

$$\begin{aligned} \gamma_{[l, m; p, 1]}^{(i)} &= (-1)^{i+1} (k-l+p) A_{[2l, 2m; 2l-p+m-2, 0]}, \\ \gamma_{[l, m; p, 0]}^{(i)} &= \frac{(-1)^i}{3} \left(\nu_i + \rho_i + lm - 3 + \frac{m(m+1)}{2} \right) A_{[2l, 2m; 2l-p+m, 2]}, \\ \gamma_{[l, m; p, -1]}^{(i)} &= (-1)^{i+1} (k+l-p) A_{[2l, 2m; 2l-p+m+2, 4]}, \\ n(\sigma_i; l, m) &= \begin{cases} (2-m)/2 & \text{if } m \in \{0, \pm 2\}, \\ (3-m)/2 & \text{if } (m, l-k) \in \{\pm 1\} \times (2\mathbf{Z}), \\ (1-m)/2 & \text{if } (m, l-k) \in \{\pm 1\} \times (1+2\mathbf{Z}), \end{cases} \end{aligned}$$

and $\delta(\sigma_i; l) \in \{0, 1\}$ such that $\delta(\sigma_i; l) \equiv l-k \pmod{2}$.

In the above equations, we put $A_{[2l, 2m; p, j]} = 0$ for $p < 0$ or $p > 2(l+m)$, and erase the symbols $\text{Diag} (f(n))$ ($c_0 > c_1$), $O_{m,n}$ ($m \leq 0$ or $n \leq 0$).

Proof. By the similarly computation in the proof of Theorem 5.5 using Lemma 6.4 (i), we obtain the assertion in the case of $i = 1$. However, in the case of $i = 2$, It is difficult to prove the assertion by the same method since the value of $T_2(l; p)$ at $1_3 \in G$ is not simple. We avoid this problem as follows.

We put

$$\begin{aligned} t'_2(l; p, j) &= \pi_{(\nu_2, \sigma_2)}(u_c) t_2(l; p, j) \quad (0 \leq j \leq 2l), \\ T'_2(l; p) &= {}^t(t'_2(l; p, 0), t'_2(l; p, 1), \dots, t'_2(l; p, 2l)), \\ \mathbf{T}'_2(\sigma_2; l) &= \begin{cases} (T'_2(l; 0), T'_2(l; 2), T'_2(l; 4), \dots, T'_2(l; l-k)) & \text{if } l-k \text{ is even,} \\ (T'_2(l; 1), T'_2(l; 3), T'_2(l; 5), \dots, T'_2(l; l-k)) & \text{if } l-k \text{ is odd,} \end{cases} \\ \mathfrak{C}'_{l,m} &= \sum_{j=0}^4 R(\iota_j^{(l,m)}) \otimes X'_j. \end{aligned}$$

Then we see that

$$(6.2) \quad \mathfrak{C}'_{l,m} \mathbf{T}'_2(\sigma_2; l) = \mathbf{T}'_2(\sigma_2; l+m) \cdot R(\Gamma_{l,m}^2),$$

and

$$T'_2(l; p)(1_3) = \mathbf{e}_{2l-p}^{(2l)} \otimes \chi_{l-p} + (-1)^l \mathbf{e}_p^{(2l)} \otimes \chi_{p-l}.$$

Thus, by the similarly computation as in Lemma 6.4 (ii), we also obtain the assertion in the case of $i = 2$ evaluating the both side of the equation (6.2) at $1_3 \in G$. \square

7. THE ACTION OF $\mathfrak{p}_{\mathbf{C}}$

The linear map $\Gamma_{l,m}^i$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. In this section, we give a explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.

7.1. The projectors for $V_l \otimes_{\mathbf{C}} V_4$. For $-2 \leq m \leq 2$, we describe a surjective $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism P_{2m}^l from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m} in terms of the standard basis as follows.

Lemma 7.1. *Let $\{v_q^{(l)} \mid 0 \leq q \leq l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\geq 0}$. We put $v_q^{(l)} = 0$ when $q < 0$ or $q > l$.*

We define linear maps $P_{2m}^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+2m}$ ($-2 \leq m \leq 2$) by

$$P_{2m}^l(v_q^{(l)} \otimes w_r) = B_{[l, 2m; q, r]} \cdot v_{q+r+m-2}^{(l+2m)},$$

when V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish.

Here the coefficients $B_{[l, 2m; q, r]} = b(l, 2m; q, r)/d'(l, 2m)$ are defined by following formulae.

Formula 1: *The coefficients of $P_4^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+4}$ are given as follows:*

$$b(l, 4; q, r) = 1 \quad (0 \leq r \leq 4), \quad d'(l, 4) = 1.$$

Formula 2: *The coefficients of $P_2^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+2}$ are given as follows:*

$$\begin{aligned} b(l, 2; q, 0) &= 4q, & b(l, 2; q, 1) &= -(l - 4q), & b(l, 2; q, 2) &= -2(l - 2q), \\ b(l, 2; q, 3) &= -(3l - 4q), & b(l, 2; q, 4) &= -4(l - q), & d'(l, 2) &= l + 4. \end{aligned}$$

Formula 3: *The coefficients of $P_0^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_l$ are given as follows:*

$$\begin{aligned} b(l, 0; q, 0) &= 6q(q - 1), & b(l, 0; q, 1) &= -3q(l - 2q + 1), \\ b(l, 0; q, 2) &= l^2 - 6lq + 6q^2 - l, & b(l, 0; q, 3) &= 3(l - 2q - 1)(l - q), \\ b(l, 0; q, 4) &= 6(l - q)(l - q - 1), & d'(l, 0) &= (l + 3)(l + 2). \end{aligned}$$

Formula 4: *The coefficients of $I_{-2}^l: V_{l-2} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:*

$$\begin{aligned} b(l, -2; q, 0) &= 4q(q - 1)(q - 2), & b(l, -2; q, 1) &= -q(q - 1)(3l - 4q + 2), \\ b(l, -2; q, 2) &= 2q(l - 2q)(l - q), & b(l, -2; q, 3) &= -(l - 4q - 2)(l - q)(l - q - 1), \\ b(l, -2; q, 4) &= -4(l - q)(l - q - 1)(l - q - 2), & d'(l, -2) &= (l + 2)(l + 1)l. \end{aligned}$$

Formula 5: The coefficients of $I_{-4}^l: V_{l-4} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} b(l, -4; q, 0) &= q(q-1)(q-2)(q-3), & b(l, -4; q, 1) &= -q(q-1)(q-2)(l-q), \\ b(l, -4; q, 2) &= q(q-1)(l-q)(l-q-1), & b(l, -4; q, 3) &= -q(l-q)(l-q-1)(l-q-2), \\ b(l, -4; q, 4) &= (l-q)(l-q-1)(l-q-2)(l-q-3), & d'(l, -4) &= (l+1)l(l-1)(l-2). \end{aligned}$$

Then P_{2m}^l is the generator of $\text{Hom}_{\mathfrak{sl}(2, \mathbf{C})}(V_l \otimes_{\mathbf{C}} V_4, V_{l+2m})$ such that $P_{2m}^l \circ I_{2m}^l = \text{id}_{V_{l+2m}}$.

Proof. The composite

$$V_l \otimes_{\mathbf{C}} V_4 \simeq V_l^* \otimes_{\mathbf{C}} V_4^* \simeq (V_l \otimes_{\mathbf{C}} V_4)^* \ni f \mapsto f \circ I_{2m}^l \in V_{l+2m}^* \simeq V_{l+2m}$$

is a surjective $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m} , which is unique up to scalar multiple. Therefore we obtain the assertion from Proposition 4.2 and Lemma 4.4. \square

7.2. The action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.

Proposition 7.2. (i) An explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{s(l; p, q) \mid l \geq 0, p \in Z(\sigma_0; l), 0 \leq q \leq 2l\}$ of $H_{(\nu_0, \sigma_0), K}$ is given by following equation:

$$\pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) = \sum_{\substack{-1 \leq j \leq 1 \\ -2 \leq m \leq 2}} \gamma_{[l, m; p, j]}^{(0)} B_{[2l, 2m; q, r]} s(l+m; p+m+2j, q+m+r-2).$$

Here we put

$$\begin{aligned} \gamma_{[0, m; 0, j]}^{(0)} &= B_{[0, 2m; 0, r]} = 0 \text{ for } m < 2, & \gamma_{[1, m; p, j]}^{(0)} &= B_{[2, 2m; q, r]} = 0 \text{ for } m < 0, \\ s(l; p, q) &= 0 \text{ whenever } p \leq l \text{ such that } p \notin Z(\sigma_0; l) \text{ or } q < 0 \text{ or } q > 2l, \\ s(l; p, q) &= (-1)^{\varepsilon(\sigma_0; l)} s(l; 2l-p, q) \text{ for } p > l. \end{aligned}$$

(ii) For $i = 1, 2$, the explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{t_i(l; p, q) \mid l \geq k, 0 \leq p \leq l-k, p \equiv l-k \pmod{2}, 0 \leq q \leq 2l\}$ of $H_{(\nu_i, \sigma_i), K}$ is given by following equation:

$$\pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) = \sum_{\substack{-1 \leq j \leq 1 \\ -2 \leq m \leq 2}} \gamma_{[l, m; p, j]}^{(i)} B_{[2l, 2m; q, r]} t_i(l+m; p+m+2j, q+m+r-2)$$

Here we put $t_i(l; p, q) = 0$ unless $0 \leq p \leq l-k, p \equiv l-k \pmod{2}$ and $0 \leq q \leq 2l$.

Proof. Since

$$\begin{aligned} \pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) &= \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^0(S(l; p)) \circ P_{2m}^l(v_q^{(2l)} \otimes X_r), \\ \pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) &= \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^i(T_i(l; p)) \circ P_{2m}^l(v_q^{(2l)} \otimes X_r) \quad (i = 1, 2), \end{aligned}$$

we obtain the assertion from Theorem 5.5, 6.5 and Lemma 7.1. \square

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